

# Vector Representation

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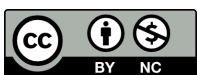
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## CONCEPT

## 1

# Vector Representation

The student will:

- explain the relationship between coordinates and components.
- use vectors and vector components to add and subtract vectors.
- use trigonometric relationships to express vector components.

## Vocabulary

- **resultant vector:** A single vector that is the vector sum of two or more other vectors.
- **trigonometric functions:** sine, cosine, tangent, inverse tangent
- **vector addition:** Vector addition is the process of finding one vector that is equivalent to the result of the successive application of two or more given vectors. Another way to define addition of two vectors is by a head-to-tail construction that creates two sides of a triangle. The third side of the triangle determines the sum of the two vectors.
- **vector components:** Parts of a vector that add up to the whole. In two-dimensional problems, there are usually two components- the horizontal x-component and the vertical y-component. Vector addition of all the components yields the original vector.
- **vector subtraction:** An operation identical to adding a negative inverse of a vector, defined as a vector in the opposite direction to the vector with the same magnitude.

## Equations

$$r = \cos \theta$$

$$y = r \sin \theta$$

$$\theta = \tan^{-1} \frac{y}{x}$$

## Introduction

The key to understanding motion in two or more dimensions is one principle: *Motion in each dimension works independently.*

The real world has three spatial dimensions. Representing motion in two dimensions can model many real world phenomena more completely than modeling in one dimension. Some examples of two-dimensional motion are:

- an arrow shot into the air
- a baseball hit into left field
- a ball rolling off a table top




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**FIGURE 1.1**

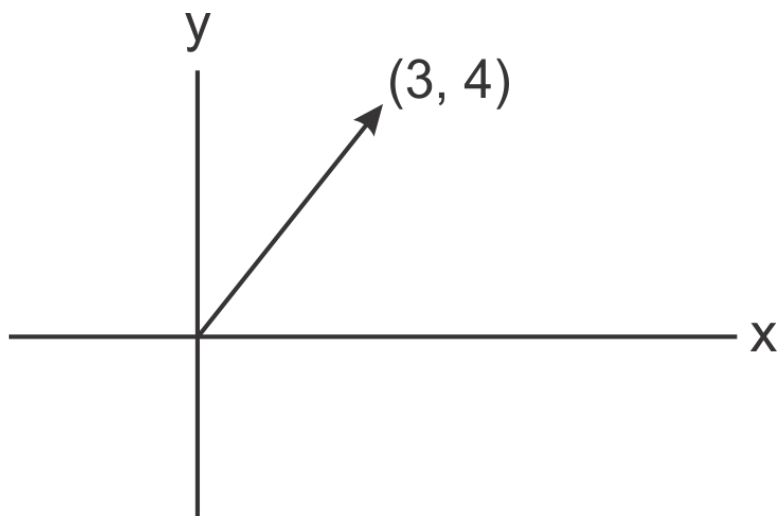
Three examples of two-dimensional motion.

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### Representing Vectors in Two Dimensions

If you were asked to give directions to the point  $(3, 4)$  in **Figure** below, you might suggest: Beginning at the origin of the coordinate system, walk along the  $x$ -axis, 3 units, then, turn left and walk 4 units in the  $y$ -direction.

The arrow, drawn from the origin to the point  $(3, 4)$  is a graphical representation of a vector quantity. By definition, a vector quantity must have magnitude (amount) and direction. The length of the arrow indicates the magnitude of the vector and the inclination of the vector to the  $x$ -axis indicates its direction. Later on, we will indicate the direction of a vector using the angle,  $\theta$ , it makes with the positive  $x$ -axis.




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**FIGURE 1.2**

A coordinate system with an arrow beginning at the origin and ending at the point  $(3,4)$ .

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When we discuss vectors, we often refer to the  $x$  and  $y$  coordinates of a point as the components of the vector; “*the  $x$ -component of  $\vec{A}$  and the  $y$ -component of  $\vec{A}$ ,*” where the bolded  $\vec{A}$  is read as, “*vector A.*” **Figure** below shows the graphical representation of the  $x$  and  $y$  components of the vector  $(3, 4)$ . The components of the vector may represent position, velocity, acceleration, force, and many other concepts.

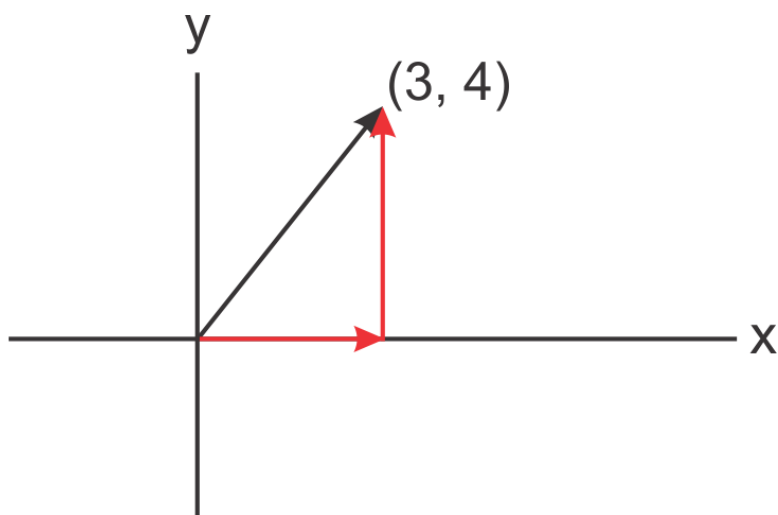


FIGURE 1.3

The red arrows represent the axial components of the vector beginning at the origin and ending at the point (3,4).

### Adding Vectors

If we move over **+3** and up **+4** as instructed above, we end up standing at the tip of the arrowhead. If we are now instructed to move **+6** more units in the  $x$ -direction and **+2** more units in the  $y$ -direction, where do we end up with respect to the origin of the coordinate system?

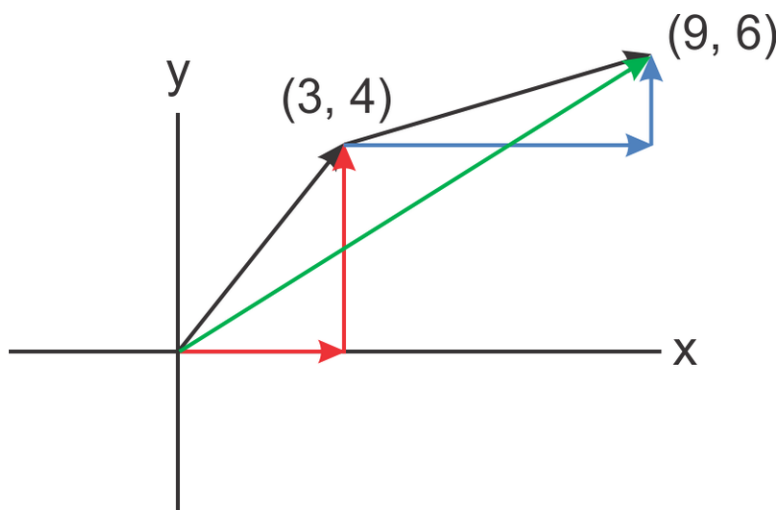


FIGURE 1.4

We can determine our location by finding the total displacement in the  $x$  and  $y$  directions, separately.  $x : (+3 + (+6)) = +9, y = (+4 + (+2)) = +6$ . Therefore, our final position is  $(+9, +6)$ . A new vector from the origin to the point  $(+9, +6)$  can now be drawn. This vector represents the sum of the two vectors and is called the **resultant vector**. (See the green vector in **Figure** above):

vector  $(+3, +4) +$  vector  $(+6, +2) =$  the resultant vector  $(+9, +6)$ .

We define vector  $(+3, +4)$  as  $\vec{A}$ , and vector  $(+6, +2)$ , as  $\vec{B}$ , and vector  $(+9, +6)$ , as  $\vec{C}$ . Thus we can state:  $\vec{A} + \vec{B} = \vec{C}$ , and consider this process as **vector addition**. If vectors  $\vec{A}$  and  $\vec{B}$  had originally been drawn from the origin (See **Figure** below), we could still add the two vectors together. Two methods to do so are presented below.

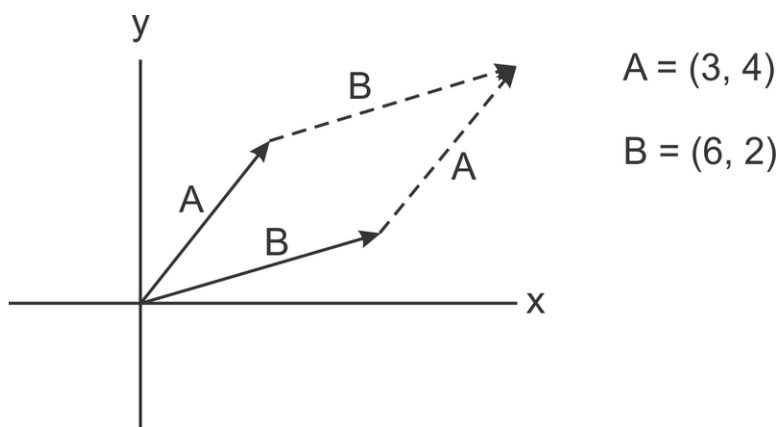


FIGURE 1.5

**Method 1:** (numerical) Add each vector's respective  $x$  and  $y$  components.

**Method 2:** (Graphical-See **Figure** above) Keep  $\vec{A}$  where it is and slide  $\vec{B}$  parallel to itself—preserve its direction and length—until the tail (the end) of  $\vec{B}$  connects to the tip (head) of  $\vec{A}$ , as we originally had done. Since the order of addition for Method 1 is irrelevant, that is:  $x = +3 + (+6)$  or  $6 + (+3)$  and  $y = +4 + (+2)$  or  $+2 + (+4)$ , this suggests that for the graphical method, the same resultant is formed regardless of whether  $\vec{B}$  is slid and  $\vec{A}$  remains where it or  $\vec{A}$  is slid and  $\vec{B}$  remains where it is. In using the graphical method an appropriate scale must be devised, such as  $1.0 \text{ cm} = 10.0 \text{ m}$  or  $1.0 \text{ cm} = 5.0 \text{ m/s}$ . Such a scale ensures that when vectors are drawn, they are correctly represented relative to each other. A protractor can also be used in order to draw the vectors correctly.

In general, both methods are valid for any number of vectors as well as for subtracting vectors.

## Vector Subtraction

**Method 1:** Subtract  $\vec{B}$  from  $\vec{A}$ : Multiply the  $x$  and  $y$  components of  $\vec{B}$  by  $(-1)$  and then add our new  $\vec{B}$  (let's call it  $-\vec{B}$ ) to  $\vec{A}$ . Thus,  $\vec{A} + (-\vec{B})$ , gives:  $x = (+3 + (-6)) = -3$ , and  $y = (+4 + (-2)) = +2$ . Recall that  $\vec{A} + (-\vec{B})$  can be written as  $\vec{A} - \vec{B}$ . Therefore,  $\vec{A} - \vec{B} = \vec{C}' = (-3, +2)$ .

**Method 2:** Since a vector can be made into its negative by multiplying its components by  $(-1)$ , the graphical transformation of this multiplication process is a vector of the same length as the original vector but directed opposite to the original vector (See **Figure** below). This is exactly what occurs when an ordered pair of coordinates has both of their signs changed. Therefore, all that is required to subtract vector  $\vec{B}$  from vector  $\vec{A}$ , is to reverse the direction of  $\vec{B}$ , and add it to  $\vec{A}$ :  $\vec{A} + (-\vec{B}) = \vec{C}$ .

<http://demonstrations.wolfram.com/SumOfTwoVectors/>

## Check Your Understanding

1. Vector  $\vec{P}$  has components  $(2, -7)$  and Vector  $\vec{Q}$  has components  $(0, -6)$ . Using the mathematical method find the sum of  $\vec{P}$  and  $\vec{Q}$ , call the resultant vector  $\vec{R}$ .

**Answer:**  $\vec{R} = (+2 + 0, -7 + -6) = (+2, -13)$ .

2. Find  $\vec{P} - \vec{Q} = \vec{R}'$ .

**Answer:**  $\vec{R}' = (+2 - 0, -7 - (-6)) = (+2, -1)$ .

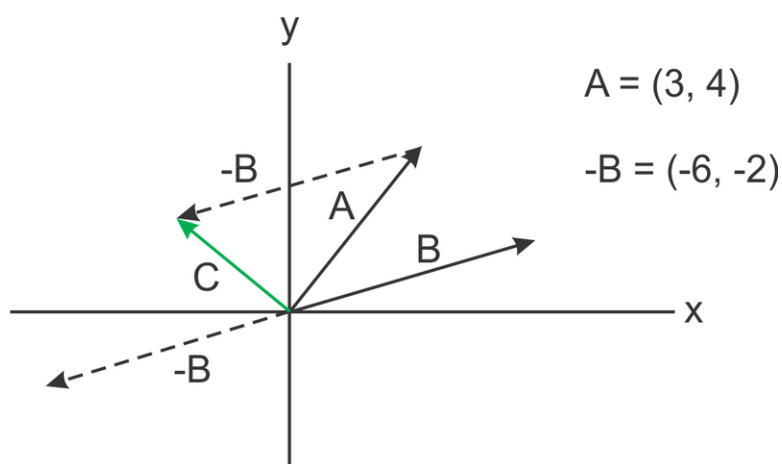


FIGURE 1.6

### Using Trigonometric Functions to Express Vector Components

Let's return to the situation we began with: finding our way from the origin to the point (3, 4). There is yet another set of directions that will take us from the origin, (0, 0) to (3, 4). Instead of moving in the  $x$  and  $y$  directions, we could move some number of units at a certain angle to the  $x$ -axis. For this particular example, moving 5 units at an angle of  $53.13^\circ$  is equivalent to walking +3 units in the  $x$ -direction, turning left, and walking +4 units in the  $y$ -direction. (Why do you think it is exactly 5 units?)

If both methods yield the same result, there must be a way to mathematically show they are identical. And there is. It can be done using a few definitions involving ratios associated with a right triangle.

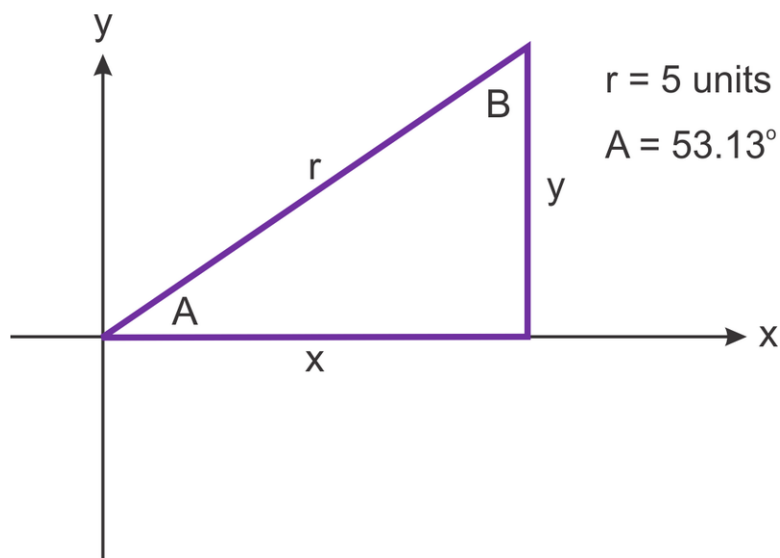


FIGURE 1.7

Angle  $A = 53.13^\circ$  and angle  $B = 36.87^\circ$

The reference angle referred to in the sine, cosine, and tangent functions is  $A$ . See **Figure** above and the definitions given below. The reference angle is *usually* the angle formed with the positive  $x$ -axis, though as we will see, angle  $B$  can just as easily be used as the reference angle.

Due to a bit of history, linguistics, and custom, we define these ratios as follows:

The **sine** ratio is  $\sin A = \frac{y}{r}$ ;  $\frac{\text{(side opposite the reference angle=y)}}{\text{(hypotenuse=r)}}$

The **cosine** ratio is  $\cos A = \frac{x}{r}$ ;  $\frac{\text{(side adjacent the reference angle=x)}}{\text{(hypotenuse=r)}}$

The **tangent** ratio is  $\tan A = \frac{\sin A}{\cos A} = \frac{y}{x}$ ;  $\frac{\text{(side opposite the reference angle)}}{\text{(side adjacent the reference angle)}}$

The reference angle is the angle opposite the  $y$  side of the triangle. Once this angle is known, it's the same angle for all the trigonometric functions. Again:  $\sin A = \frac{y}{r}$ ,  $\cos A = \frac{x}{r}$  and  $\tan A = \frac{y}{x}$ .

### Check Your Understanding

1. There are two acute angles in a right triangle. What ratios define sine, cosine, and tangent of angle  $B$  in the diagram above?

**Answer:**  $\sin B = \frac{x}{r}$ ,  $\cos B = \frac{y}{r}$  and  $\tan B = \frac{x}{y}$

2.  $A = 53.13^\circ$  and  $c = 5.00$  units. Use the above definitions to verify the lengths of two legs of the right triangle.

**Answers:**  $x$ -leg: Using the definition of the cosine:  $\cos(53.13^\circ) = \frac{x}{5}$ ,  $\rightarrow 5 \cos(53.1^\circ) = x = 3.00$

$y$ -leg: Using the definition of the sine:  $\sin(53.13^\circ) = \frac{y}{5}$ ,  $\rightarrow 5 \sin(53.1^\circ) = y = 3.999 = 4.00$ .

3. Use angle  $B$  to do the same:

**Answers:**  $x$ -leg: Using the definition of the sine:  $\sin(36.87^\circ) = \frac{x}{5}$ ,  $\rightarrow 5 \sin(36.87^\circ) = x = 3.00$

$y$ -leg: Using the definition of the cosine:  $\cos(36.87^\circ) = \frac{y}{5}$ ,  $\rightarrow 5 \cos(36.87^\circ) = y = 4.00$

In general, when the reference angle is along the  $x$ -axis:  $x = r \cos \theta$  and  $y = r \sin \theta$ ; and the ordered pair  $(x, y) = (r \cos \theta, r \sin \theta)$ . A notation used to represent a vector in terms of an angle (direction) and a length (magnitude), is  $(r, \theta)$ . Using the numerical values in the last example we have  $(5, 53.13^\circ)$ . We interpret  $(5, 53.13^\circ)$ , to mean the vector makes a positive  $53.13$  degree angle, when rotated counterclockwise from the positive  $x$ -axis, and has a length of 5 units from the origin to the tip of the arrowhead.

4. Given:  $\vec{C} = (10.0, 30^\circ)$  and  $\vec{D} = (20.0, 60^\circ)$ , find the components of  $\vec{R}$ , where  $\vec{R} = \vec{C} + \vec{D}$ .

**Answer:**  $C_x + D_x = 10 \cos(30^\circ) + 20 \cos(60^\circ) = 18.7$

$C_y + D_y = 10 \sin(30^\circ) + 20 \sin(60^\circ) = 22.3$

$\vec{R} = (18.7, 22.3)$

25 What is the magnitude of  $\vec{R}$ ?

**Answer:** Use the Pythagorean formula:  $\sqrt{(18.7^2 + 22.3^2)} = 29.1$

Magnitude of  $R = 29.1$

6. Find the direction of  $\vec{R}$  as measured counterclockwise from the positive  $x$ -axis.

**Answer:**  $\vec{R}$  is in the first quadrant, since both  $x$  and  $y$  are positive. Hence, we know the angle must be between 0 and 90 degrees. Furthermore, since  $y$  is larger than  $x$ , the angle must be greater than 45 degrees. Using the ratio associated with the tangent function we have:

$\tan \theta = \frac{22.32}{18.66}$

Therefore, we need to find the angle which gives the ratio  $\frac{22.32}{18.66}$ . This can be done using the inverse function on your calculator.  $\theta = \tan^{-1} 1.196 = 50.10^\circ$ . Finally,  $\vec{R} = (29.09, 50.10^\circ)$ .



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## References

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