Inclined Planes

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CONCEPT 1 Inclined Planes

The student will:

- Understand how to analyze and work with forces on inclined planes
- Understand how to apply Newton's Second Law to the inclined plane problems

Vocabulary

Inclined plane

Introduction

An "inclined plane" just means any flat surface tilted somewhere between horizontal and vertical, like a ramp, a flat side of a hill, or a playground slide. Problems using inclined planes shows us how to divide up vector forces including gravity, normal force, and friction.

Using Inclined Planes

For objects in motion along inclined planes it is convenient to establish the $x-$axis along the plane. An object undergoing motion along an inclined plane has at least two forces acting upon it: the force of gravity and the normal force from the inclined plane.

As we will see, the normal force is always less than the weight, when the object is placed on an inclined plane. To understand this, consider two extremes for an inclined plane:

1. A horizontal inclined plane ($\theta = 0$ degrees)
2. A vertical inclined plane ($\theta = 90$ degrees).

In the first case, the object is at rest because the net force on it is zero. In the second case the object has a net force on it of mg and therefore accelerates at g.

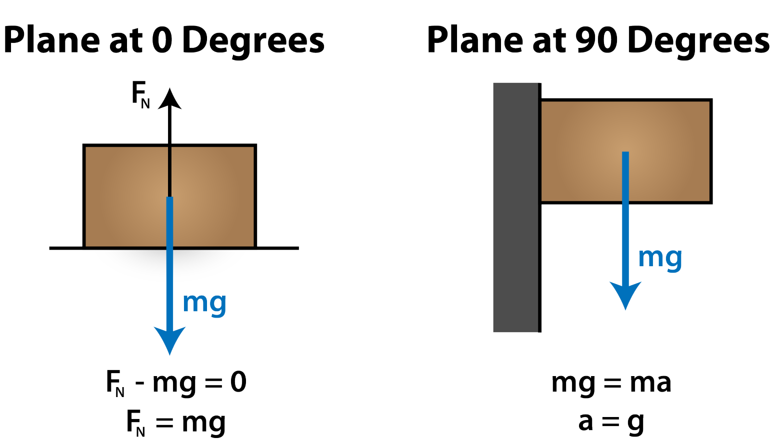

FIGURE 1.1
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Figure below shows an example between the two extremes. An object of weight 100 N is sliding down a frictionless 30-degree inclined plane. We expect the acceleration of the object to be between zero and g, and the normal force to be between 0 and 100 N.

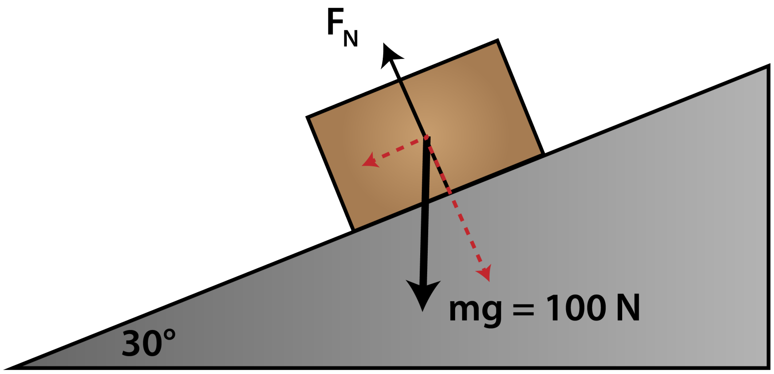

FIGURE 1.2
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Consider the following questions.

1a. What normal force does an inclined plane provide in supporting a 100 N crate accelerating down a 30-degree inclined plane?

1b. What acceleration, in the absence of friction, does the crate have along the inclined plane?

1c. What is the velocity of the crate after 3 seconds?

1d. What is the displacement of the crate after 3 seconds?

In order to answer these questions, we establish a coordinate system such that the $x-$axis is along the inclined plane and the $y-$axis is perpendicular to the plane. Next, we construct a Free-Body-Diagram (FBD) for the crate.

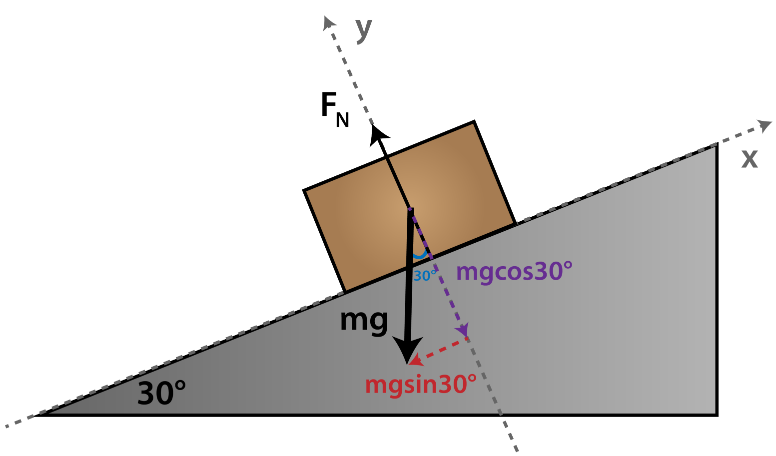

FIGURE 1.3
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What to notice:

1. The components of mg in the diagram above are dashed.
2. The angle of the incline is the same as the angle between the $y-$axis and mg.
3. The $x-$component of mg is $mg\sin 30^\circ$ and the $y-$component of mg is $mg\cos 30^\circ$.

We use Newton's Second Law in the x and y directions in order to answer questions 1 and 2.

1. To determine the normal force that the inclined plane provides we use NSL for the forces in the $y-$direction.

$$\begin{aligned}\sum F_y &= F_N - mg\cos\theta = 0\\ F_N &= mg\cos\theta\\ F_N &= 100\text{N}\cos 30^\circ\\ F_N &= 100\text{N}(0.87) = 87\text{ N}\end{aligned}$$

2. To determine the acceleration of the crate down the plane we use Newton's Second Law for the forces in the $x-$direction.

$$\begin{aligned}\sum F_x &= mg\sin\theta = ma,\\ a &= g\sin\theta,\\ a &= (10\text{m/s}^2)\sin 30^\circ = 5.0\text{ m/s}^2\end{aligned}$$

3. To determine the velocity of the crate at $t = 3.0$ s we use the fact $a = 5.0$ m/s^2.

Every second the crate gains another 4.9 m/s of velocity so after 3.0 s: $(5.0\text{m/s}^2)(3.0\text{s}) = 15$ m/s

In other words, $v_f = at + v_i$, where the initial velocity is zero.

4. To determine the displacement of the crate at $t = 3.0\ s$ we can use either acceleration or average velocity. For the former, $a = 5.0$ m/s^2 for the latter, $\frac{Vi+Vf}{2} = \frac{0+15}{2} = 7.5$ m/s,

Using the acceleration:

$$\begin{aligned}x_f &= \frac{1}{2}at^2 + v_it + x_i,\\ x_f &= \frac{1}{2}(5.0\text{m/s}^2)(3.0\text{s})^2 + 0 + 0 = 22.5\text{ m}\end{aligned}$$

Using the average velocity: $x = v$

avgt = (7.5m/s)(3.0s) = 22.5 m

http://demonstrations.wolfram.com/BlockOnAFrictionlessInclinedPlane/

Check your understanding

1. True or False? The direction of the normal force the inclined plane exerts on an object is opposite to the direction of mg.

Answer: False. The normal force is perpendicular to the inclined plane.

2. True or False? The weight of a crate is 200 N. Its weight on an inclined plane will be smaller than 200 N.

Answer: False, its weight is always 200 N.

3. True or False? The weight of a crate is 200 N. The normal force exerted on it when placed on an inclined plane will be smaller than 200 N.

Answer: True. The normal force is $F_N = mg\cos\theta$. Since the cosine function decreases as the angle increases, the normal force decreases as well.

3. What is the acceleration of an object sliding down a frictionless inclined plane?

Answer: $a = g\sin\theta$

Friction and Inclined Planes

There's a quick activity that can be used to show the effect of static friction on an inclined plane. Get a hold of a book and a coin (See picture below). Place the coin on the top of the cover of the book and slowly begin to lift the cover. You'll notice that as you raise the cover the coin stays put. If the cover were a frictionless surface, the coin would slide down immediately as you began to lift the cover. Static friction keeps the coin stationary. However, there is a point at which the coin will begin to slide. There's a critical height (angle) at which the component of the force of gravity down the inclined cover exceeds the force of static friction. If you play with this set-up for a while, you'll notice that once the coin begins to slide you can slightly lower the cover and the coin continues sliding even though the cover is raised less than to the extent that was needed to have the coin begin its slide. This is another reminder that static friction tends to be greater than kinetic friction. We like to quantity how static and kinetic friction affects objects on inclined planes. We will take the case of static friction first.

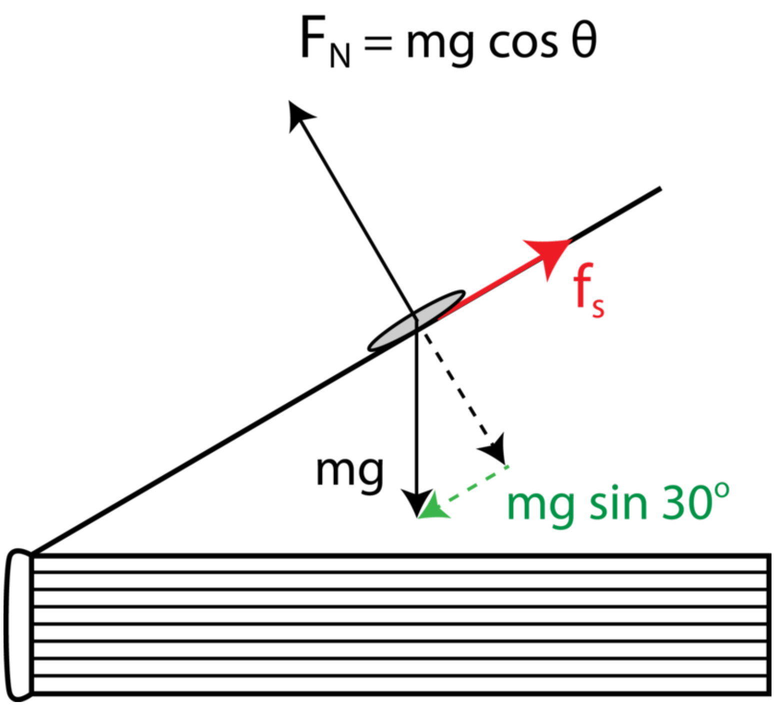

FIGURE 1.4
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Figure above shows the coin at the point where it is just about to slide. Up to this point static friction, f_s, is equal and opposite to the component of force due to gravity down the inclined plane, $mg\sin\theta$. We can use Newton's Second Law to solve for the coefficient of static friction μ_s.

$$\begin{aligned}
&\sum_x F = f_s - mg\sin\theta = 0,\ \textit{replacing}\ f_s\ \textit{with}\ \mu_s F_N,\\
&\mu_s F_N * mg\sin\theta = 0,\ \textit{replacing}\ F_N\ \textit{with}\ mg\cos\theta,\\
&\mu_s(mg\cos\theta) - mg\sin\theta = 0,\\
&\mu_s = \frac{\sin\theta}{\cos\theta},\\
&\mu_s = \tan\theta
\end{aligned}$$

Check your understanding

1. A coin rests on a 30-degree inclined plane. It is determined that if the plane's angle is increased further, the coin will slide down the plane. What is the coefficient of static friction between the coin and the plane?

Answer: Since $\mu_s = \tan\theta, \mu_s = \tan 30^\circ = 0.577, \mu_s = 0.58$

2. True or False? Since the friction force is dependent upon the normal force, the coefficient of friction must have units of newtons.

Answer: False. As you can see from the inclined plane problem, the coefficient of friction is dependent only upon the tangent of the angle. Recall that the definition of μ_s is the ratio of the friction force to the normal force and therefore as no units (it is a pure number).

If we exceed the maximum angle the *static* friction can hold the coin, the coin will slide and *kinetic* friction will act upon the coin. We consider this situation next. **Figure** below is almost identical to **Figure** above. The only differences are that the coin is represented as a dot and the kinetic friction vector has been drawn a bit smaller than the static friction vector.

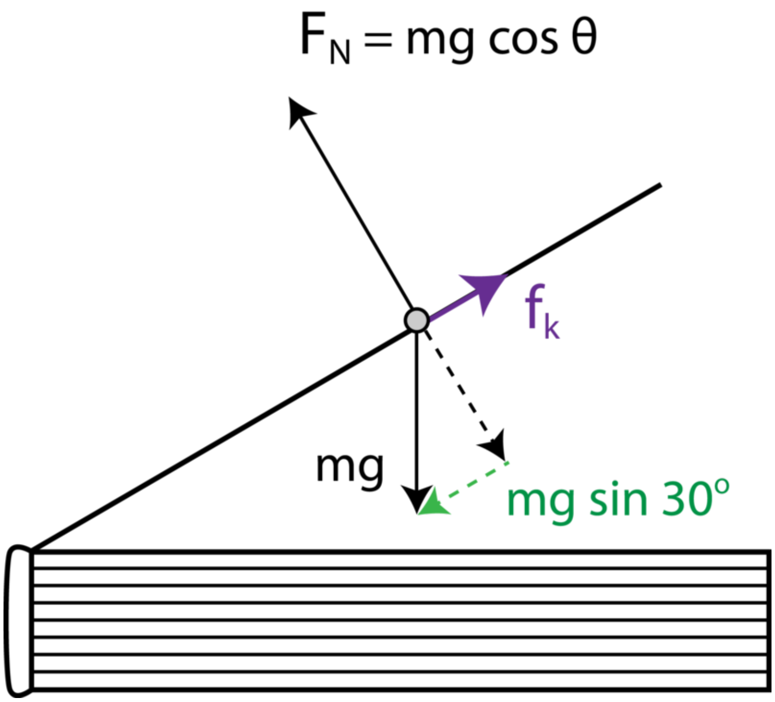

FIGURE 1.5

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In this case, the coin will accelerate down the book cover. We wish to determine the acceleration. We choose the direction down the book cover as positive. Using Newton's Second Law:

$$\Sigma F_x = mg\sin\theta - f_k = ma_x$$

, replacing f_k with

$$\mu_k F_N$$

$$mg\sin\theta - \mu_k F_N = ma_x$$

, replacing F_N with

$$mg\cos\theta$$

$$mg\sin\theta - \mu_k(mg\cos\theta) = ma_x$$

$$a_x = g(\sin\theta - \mu_k\cos\theta)$$

Check your understanding

1. A block slides down a 25-degree incline with a constant velocity. What is the coefficient of kinetic friction between the block and the incline?

Answer: $\mu_k = 0.466 = 0.47$. Hint: Consider the result of the last example.

2a. A skier of mass 60 kg skies down a slope of 18 degrees with an acceleration of 2.1 m/s^2. What is the force of friction on the skis?

Answer:

$$\begin{aligned}\sum F &= mg\sin\theta - f_k = ma\\ f_k &= mg\sin\theta - ma \rightarrow (60)(9.8)\sin 18^\circ - (60)(2.1)\\ f_k &= 55.7 = 56\ N\end{aligned}$$

2b. What is μ_k?

Answer: $f_k = \mu_k F_N = \mu_k(60)(9.8)\cos 18^\circ = 55.7\ N$ and therefore $\mu_k = 0.099 = 0.10$

2c. The skier is hurt and a member of a rescue team pulls the skier up the slope at a constant velocity.

(i) What force must he exert?

(ii) What percentage of the person's weight is this force?

Answers:

(i) $F = (60)(9.8)\sin 18^\circ + 0.099(60)(9.8)\cos 18^\circ = 237 = 240\ N$

(ii) $W = mg = (60)(9.8) = 588 = 590\ N \rightarrow ratio = \frac{F}{W} = \frac{237}{588} = 0.403 = 40\%$

2d. What is one benefit of using an inclined plane to lift heavy objects?

Answer: A smaller force is usually required to move objects up the plane than lifting against gravity.

An Incline Plane-Pulley System

An illustrative example

How do we determine the acceleration and the tension between two masses when one of the masses is on the inclined plane and the other hangs over the plane, as in **Figure** below?

Again, we call on Newton's Second Law in order to solve the system. Let's assume there is no friction in the system and the rope between the masses does not stretch and has negligible mass. The values for the masses and the angle of the incline are given below.

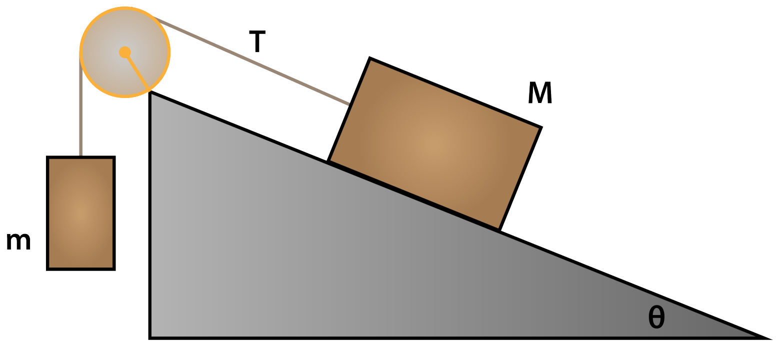

FIGURE 1.6
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$M = 40.0\ kg$

$m = 30.0\ kg$

$\theta = 25.0°$

Find, a. the acceleration of the system and, b. the tension in the rope.

Part a. We always start by drawing FBDs and resolving vectors.

1. Draw a FBD for each mass.

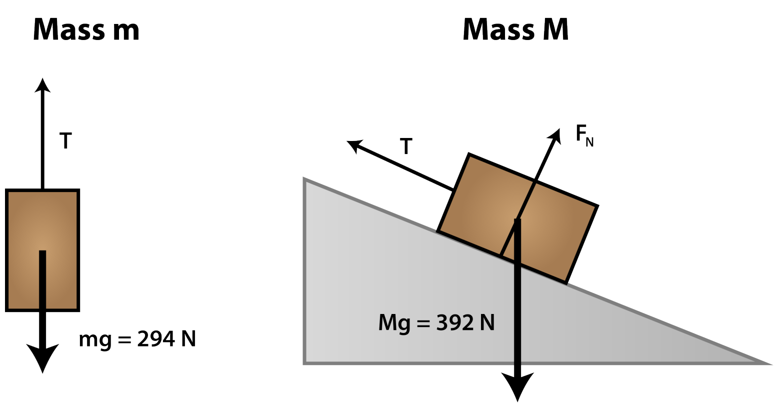

FIGURE 1.7
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2. Resolve any vectors that are not along the x or y axes.

The $x-$component of Mg: $Mg \sin\theta$

The $y-$component of Mg: $Mg \cos\theta$

Newton's Second Law equation for the mass m:

We assume that mg moves down and therefore $mg > T$. (This is an arbitrary choice.)

$\sum_m F = mg - T = ma; \rightarrow \sum_m F = (30\text{kg})(10\text{m/s}^2) - T = 30a$

Newton's Second Law equation for M

Since m moves down M moves up the incline therefore $T > Mg$.

$\sum_M F = T - Mg\sin\theta = Ma \rightarrow \sum_M F = T - (40)(9.8)\sin 25° = 40a$

Solving the two equations above simultaneously for the acceleration gives:

$(30)(9.8) - T + T - (40)(9.8)\sin 25° = 70a; a = 1.83 = 1.8\ m/s^2$

Had the answer come out negative it would have indicated that we had chosen the direction of the acceleration incorrectly. It is also helpful to note that the tensions, T and $-T$, add to zero. The tensions T and $-T$ are a Newton's Third Law (N3L) pair of forces. Such a pair cannot act on the same object. The "object" in this case is the system. From the point of view of the system a Newton's Third Law pair of forces is considered "internal" to the system. Internal forces must always add to zero when a simultaneous set of Newton's Second Law equations are solved. The only forces remaining (on the left side of the equation) are the "external" forces acting on the system.

b. Find the tension between the two masses.

Since the acceleration is known, we can substitute its value in either the Newton's Second Law equation for the mass m or the mass M. Using the equation for the mass m is a bit simpler so:

$T = (30)(9.8) - (30)(1.83) = 239.1 = 240\ N$

As a check for reasonableness we can determine if the tension is greater than $Mg\sin\theta$ and less than mg. Try it and see!

Check your understanding

1. True or False? In order for a hanging mass M_1 to move a mass M_2 up along an inclined plane, M_1 must be greater than M_2.

Answer: False. As we saw in the last problem, m was smaller than M and M still moved up the plane.

There is, however, a "special" value of the mass m which can keep the system stationary or have the system move up or down the plane with constant velocity! We consider this situation in question 2.

2. For a given mass M, what value of the mass m would keep the system at rest or moving with a constant velocity?

We go back to the equations for m and M in the last problem and assume the acceleration is zero. Simplifying, we have:

$mg - Mg\sin\theta = 0$

Thus $m = M\sin\theta$ is the condition for which the system remains at rest or moves with constant velocity. A mass m of 16.9 kg would satisfy this condition in the last problem. (Can you see why?)

A Pulley System with Friction along the Inclined Plane

Illustrative Example

Refer back to **Figure** above. Assume that friction exists between the mass $M = 40\ kg$ and the plane. A diagram of this situation is identical to **Figure** above except that friction is included opposing the motion of M. (See **Figure** above). Let's find the acceleration of the system and the tension in the rope for this problem. We'll assume that the mass $m = 30\ kg$ accelerates down the plane.

Since friction is directed along the plane, it has only an $x-$component, directed opposite the tension T and in the same direction as $Mg\sin\theta$. The Newton's Second Law equation for the hanging mass m remains the same. Let's assume that $\mu_k = 0.20$ and keep all other values the same as in the last problem.

a. Find the acceleration of the system.

Newton's Second Law m:

$$\sum_m F = (30)(9.8) - T = 30a\ Equation\ m$$

NSL M:

$$\sum_M F = T - (40)(9.8)\sin 25^\circ - f_k = 40a,\ \textit{replacing}\, f_k\ \textit{with}\ \mu_k F_N,$$
$$T - (40)(9.8)\sin 25^\circ - \mu_k F_N = 40a,\ \textit{replacing}\ F_N\ \textit{with}\ mg\cos\theta,$$
$$T - (40)(9.8)\sin 25^\circ - 0.20(40)(9.8)\cos 25^\circ = 40a\ \textit{Equation}\ M$$

Solving Equations m and M simultaneously we find, $a = 0.818 = 0.82\ m/s^2$.

b. Find the tension.

Using Equation m we have $T = 269 = 270\ N$

Check your understanding

In the absence of friction, if $m = M\sin\theta$, the system may either move with constant velocity or remain stationary. If kinetic friction is present, is it possible for the system to move with constant velocity if $m \neq M\sin\theta$?

Answer: Yes. Try solving the equations with the acceleration equal to zero but keep μ_k as an unknown. The value that you find for μ_k will give the coefficient of friction necessary to keep the system moving with constant velocity while m and M remain 30 kg and 40 kg, respectively. If you've done the work correctly, you should find $\mu_k = 0.47$.